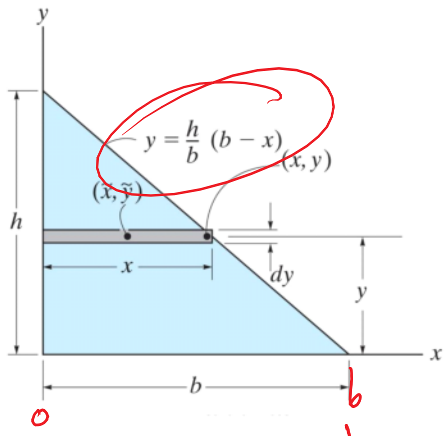
Find the centroid of the area below



find $\bar{x}$

$y = \frac{h}{b}(b - x)$

$y = h\left(1 - \frac{x}{b}\right)$

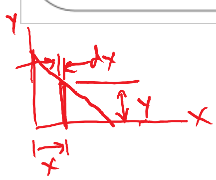
$$\bar{x} = \frac{\int x \cdot dA}{\int dA}$$

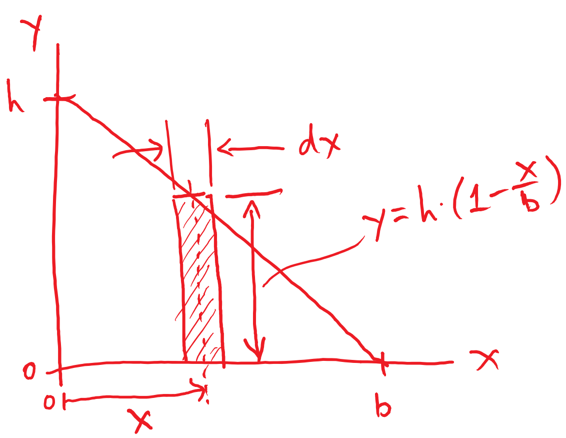
$$\int dA = \int_0^b y \cdot dx = \int_0^b h \cdot \left(1 - \frac{x}{b}\right) dx$$

$$= h \int_0^b \left(1 - \frac{x}{b}\right) dx = h\left(x - \frac{x^2}{2b}\right)\Big|_0^b$$

$$= h \cdot \left(b - \frac{b^{\cancel{2}}}{2\cancel{b}}\right) = h \cdot \frac{b}{2}$$

$$= \frac{1}{2} \cdot b \cdot h$$ Area of a triangle





Area of the trapezoid is $dA = y \cdot dx$

Centriod of the trapezoid is at $\tilde{x} = x$.

$$\Rightarrow \int \tilde{x} \cdot dA = \int_0^b x \cdot y \cdot dx$$

$\bar{x} = \int x\, dA$ $\quad \int_0^b x \cdot h \cdot \left(1 - \frac{x}{b}\right) dx$ $\quad 2h \int^b \left(x - \frac{x^2}{b}\right) dx$

$$\Rightarrow \int \tilde{x}\cdot dA = \int_0^b x\cdot y\cdot dx$$

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^b x\cdot h\cdot\left(1-\frac{x}{b}\right)dx}{\frac{1}{2}bh} = \frac{2\cancel{h}}{b\cancel{h}}\int_0^b\left(x-\frac{x^2}{b}\right)dx$$

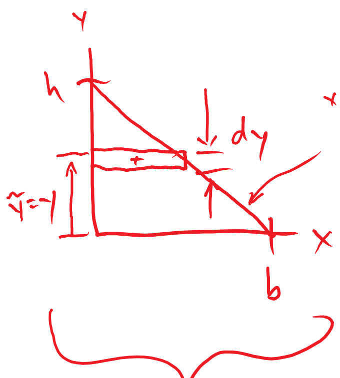
$$= \frac{2}{b}\left(\frac{x^2}{2}-\frac{x^3}{3b}\right)\Big|_0^b$$

$$= \frac{2}{b}\left(\frac{b^2}{2}-\frac{b^{\cancel{3}\,2}}{3\cancel{b}}\right) = \frac{2}{\cancel{b}}\left(\frac{3b^{\cancel{2}}}{6}-\frac{2b^{\cancel{2}}}{6}\right)$$

$$= \frac{2b}{6} = \frac{b}{3}$$

$$\boxed{\bar{x} = \frac{b}{3}}$$

Similarly, it can be shown that $\boxed{\bar{y} = \frac{h}{3}}$



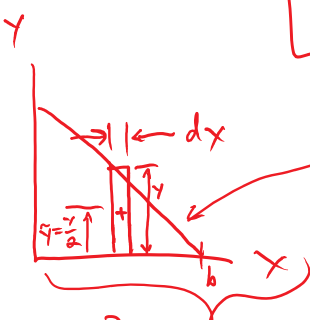
$y = h\cdot\left(1-\frac{x}{b}\right)$

$x = -b\left(\frac{y}{h}-1\right)$

$x = b\cdot\left(1-\frac{y}{h}\right)$

$\int_0^h y\cdot \underbrace{x\cdot dy}_{dA}$

$\int_0^h b\cdot\left(y-\frac{y^2}{h}\right)dy$



$y = h\cdot\left(1-\frac{x}{b}\right)$

$\tilde{y} = y$

What is $\tilde{y}$?

$\tilde{y} = \frac{y}{2}$

$$\bar{Y} = \frac{\int \tilde{y}\cdot dA}{\int dA} = \frac{\int_0^b\left(\frac{y}{2}\right)\cdot \overbrace{y\cdot dx}^{dA}}{\frac{1}{2}b\cdot h}$$

$\overbrace{\qquad\qquad}^{y^2}$ b

$$\int dA \qquad\qquad \tfrac{1}{2} b \cdot h \qquad\qquad y^2$$

$$= \frac{\int_0^b \cancel{\tfrac{1}{2}} y^2 \cdot dx}{\cancel{\tfrac{1}{2}} \cdot b \cdot h} = \frac{1}{bh} \int_0^b \overbrace{h^2 \cdot \left(1 - \frac{x}{b}\right)^2}^{y^2} dx$$

$$= \frac{h}{b} \int_0^b \left(1 - \frac{2x}{b} + \frac{x^2}{b^2}\right) dx$$

$$= \frac{h}{b} \left(x - \frac{x^2}{b} + \frac{x^3}{3b^2}\right)\Big|_0^b$$

$$= \frac{h}{b} \left(\cancel{b} - \frac{\cancel{b^2}}{\cancel{b}} + \frac{b^{\cancel{3}}}{3\cancel{b^2}}\right)$$

$$= h\left(1 - 1 + \frac{1}{3}\right)$$

$$\boxed{\bar{Y} = \frac{h}{3}}$$

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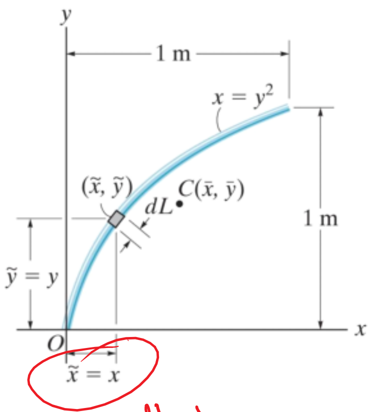
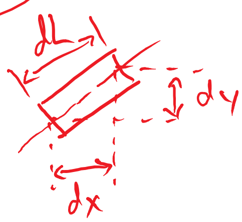
Find the centroid of the rod bent into the shape of a parabolic arc





$$\bar{x} = \frac{\int \tilde{x} \cdot dL}{\int dL}$$

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{(dx)^2 \cdot \left[1 + \frac{(dy)^2}{(dx)^2}\right]}$$

$$= dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$L = \int dL = \int_0^{1m} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$x = y^2 \qquad y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$L = \int_0^{1m} \sqrt{1 + \frac{1}{4x}} \cdot dx$$

$$L = 1.4289 m$$

$$\bar{x} = \frac{\int \tilde{x} \cdot dL}{L} \qquad \tilde{x} = x$$

$$\boxed{\bar{x} = \frac{\int_0^{1m} x\sqrt{1 + \frac{1}{4x}}\, dx}{1.4289 m} = \frac{0.6063 \, m^2}{1.4789 \, m} = 0.410 m}$$

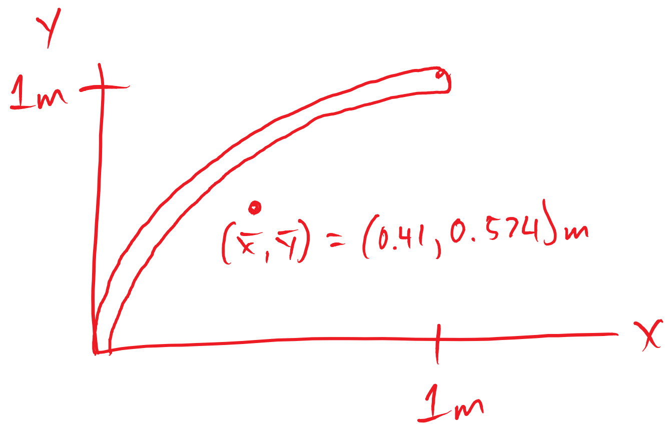
same setup for $\bar{y}$:

$$\bar{y} = \frac{\int_0^{1m} y \cdot \sqrt{1 + \frac{1}{4x}} \cdot dx}{1.4789m} = \frac{\int_0^{1m} \sqrt{x} \sqrt{1 + \frac{1}{4x}} \cdot dx}{1.4789m}$$

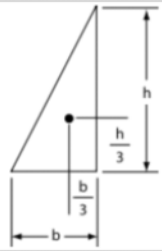
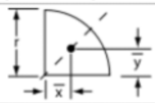
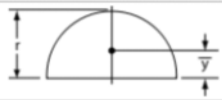
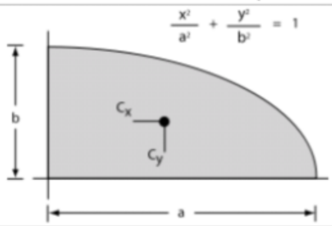
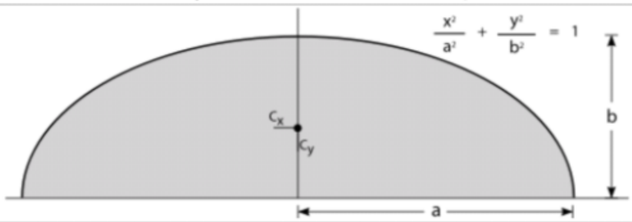
$\vdots$

$$\bar{y} = 0.574 m$$

$\bar{y} = 0.574\,m$



Centroid of typical 2D shapes

Shape	Figure	$\bar{x}$	$\bar{y}$	Area
Right-triangular area		$\frac{b}{3}$	$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$

http://en.wikipedia.org/wiki/List_of_centroids

Applications

The I-beam (top) or T-beam (bottom) shown are commonly used in building various types of structures.

How can we easily determine the location of the centroid for different beam shapes?

centroid of dA

centroid formula: $\bar{x} = \dfrac{\int \tilde{x} \cdot dA}{\int dA}$

integrals work great for continuous functions

discretize the integral:

$$\bar{x} \cdot \int dA = \int \tilde{x} \cdot dA$$

$$\bar{x} \cdot \sum_i A_i = \sum_i (\bar{x}_i \cdot A_i)$$

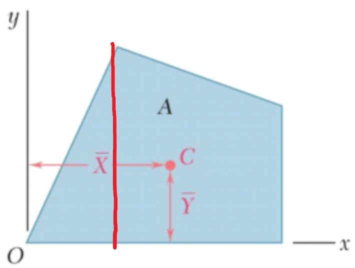
For a composite body, the centroid can be found by:

$$\bar{x} = \frac{\sum_i (\bar{x}_i \cdot A_i)}{\sum_i A_i}$$

$\bar{x}_i$ is the x-coordinate of the centroid of area A_i

Composite bodies

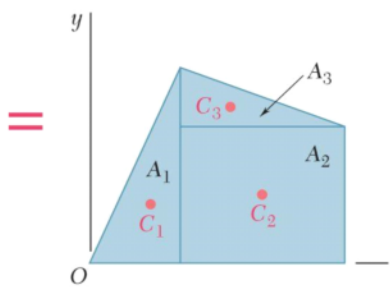
A composite body consists of a series of connected simpler shaped bodies. Such body can be sectioned or divided into its composite parts and, provided the weight and location of the center of gravity of each of these parts are known, we can then eliminate the need for integration to determine the center of gravity of the entire body.



For example, the centroid of the area A is located at point C of coordinates $\bar{x}$ and $\bar{y}$. In the case of a composite area, we divide the area A into parts A_1, A_2, A_3

$$A_{total}\,\bar{X} = \sum_i A_i\,\bar{x}_i$$

$$A_{total}\,\bar{Y} = \sum_i A_i\,\bar{y}_i$$



$$\bar{X} = \frac{A_1 \cdot \bar{X}_1 + A_2 \cdot \bar{X}_2 + A_3 \cdot \bar{X}_3}{A_1 + A_2 + A_3}$$

$$\bar{Y} = \frac{A_1 \cdot \bar{Y}_1 + A_2 \cdot \bar{Y}_2 + A_3 \cdot \bar{Y}_3}{A_1 + A_2 + A_3}$$

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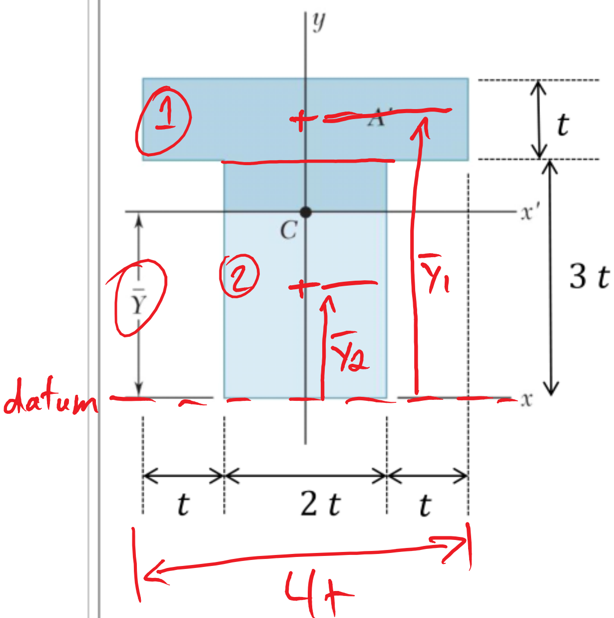
Find the centroid of the area below.



By symmetry, $\bar{x} = 0$

$$\bar{Y} = \frac{\sum \bar{y}_i A_i}{\sum A_i} = \frac{\bar{y}_1 \cdot A_1 + \bar{y}_2 A_2}{A_1 + A_2}$$

$\bar{y}_1 = 3t + \frac{1}{2}t = 3.5t$

$A_1 = (4t)\cdot t = 4t^2$

$\bar{y}_2 = 1.5 \cdot t$

$A_2 = (2t)(3t) = 6t^2$

$$\boxed{\bar{Y} = \frac{(3.5t)(4t^2) + (1.5t)(6t^2)}{4t^2 + 6t^2} = \frac{14t^3 + 9t^3}{10t^2} = 2.3 \cdot t}$$

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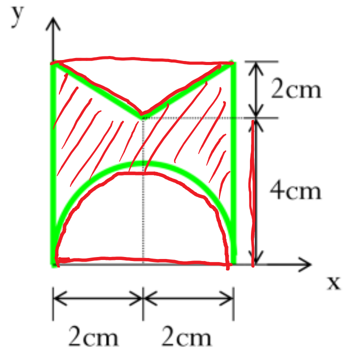
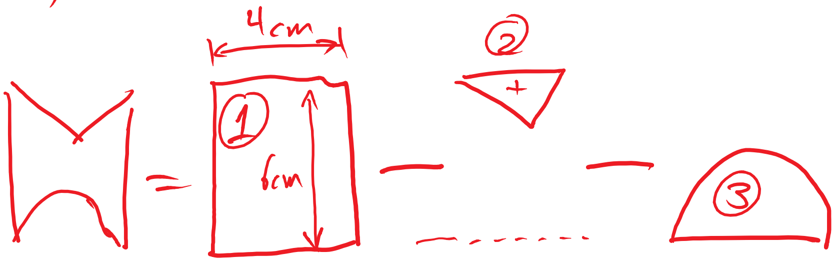
A rectangular area has semicircular and triangular cuts as shown. What is the centroid of the resultant area?



$\bar{X} = 2\text{ cm}$



$\bar{y}_1 = 3\text{cm}$

$A_1 = 24\text{ cm}^2$

$\bar{y}_2 = 4\text{cm} + \frac{2}{3}(2\text{cm})$

$\bar{y}_2 = 5.\overline{33}\text{ cm}$

$A_2 = \frac{1}{2}(4\text{cm})(2\text{cm})$

$= 4\text{ cm}^2$

$\bar{y}_3 = \frac{4(2\text{cm})}{3\pi} = \frac{8}{3\pi}\text{cm}$ $\qquad y_a = \frac{4r}{3\pi}$

$A_3 = \frac{\pi r^2}{2} = \frac{\pi(2\text{cm})^2}{2}$

$= 2\pi \cdot \text{cm}^2$

$$\bar{Y} = \frac{\bar{y}_1 A_1 - \bar{y}_2 A_2 - \bar{y}_3 A_3}{A_1 - A_2 - A_3} = \frac{(3\times 24) - (5.\bar{3}\times 4) - (\frac{8}{3\pi}\, 2\pi)}{24 - 4 - 2\pi}\text{ cm}$$

$$\boxed{\bar{Y} = \frac{68}{3(10-\pi)}\text{ cm} \qquad \bar{Y} \approx 3.305\text{ cm}}$$

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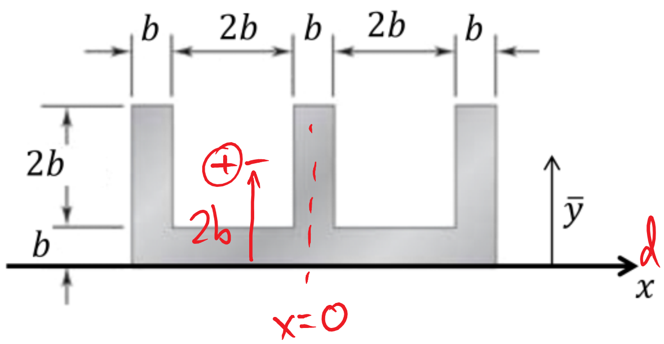
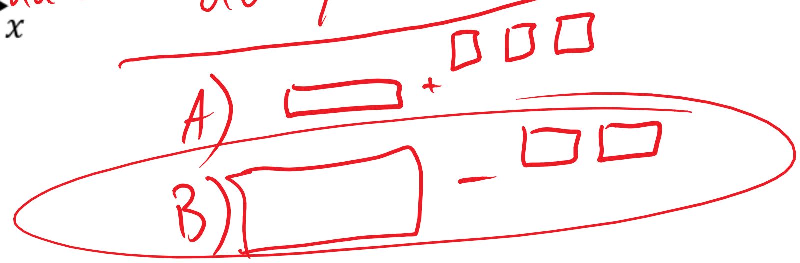
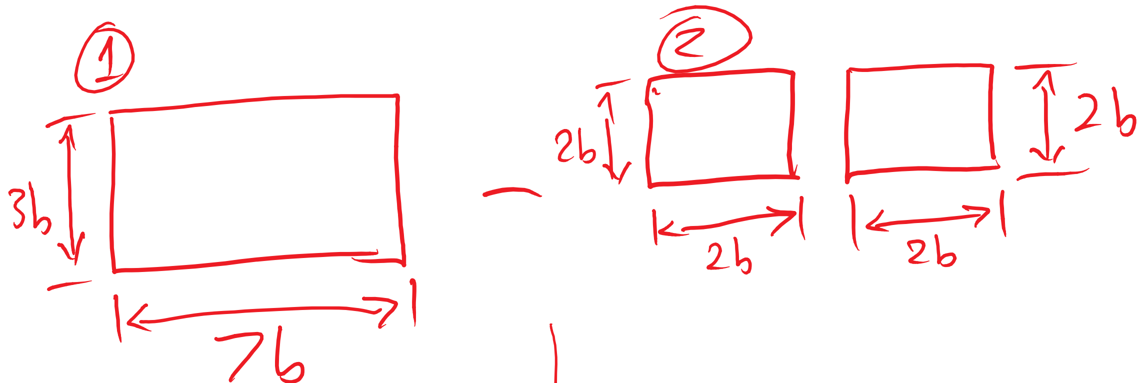
Find the centroid of the area below.



$\bar{x} = 0$

Which approach do you prefer?





$A_1 = 21 b^2$

$\bar{y}_1 = 1.5\, b$

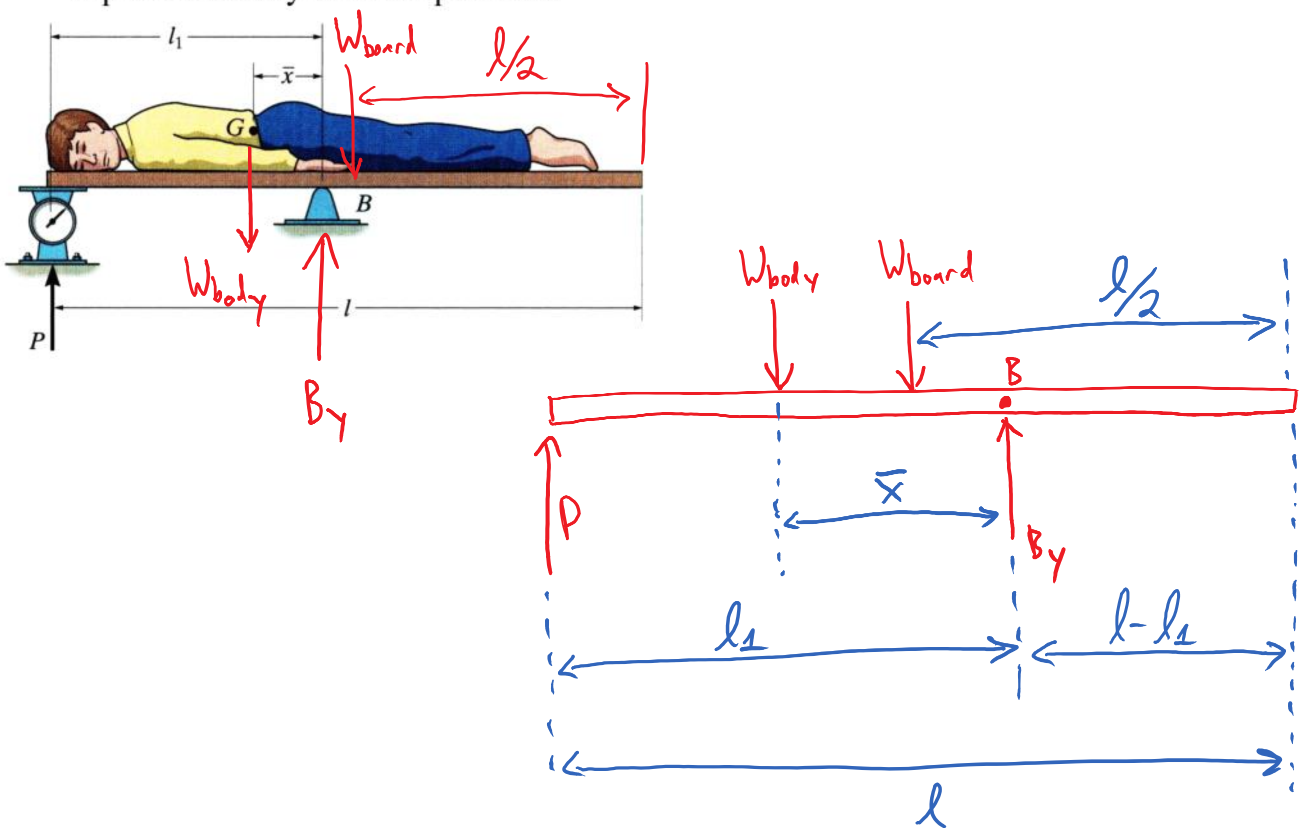
$A_2 = (2b)^2 = 4b^2$

$\bar{y}_2 = b + b = 2b$

$$\bar{Y} = \frac{A_1 \bar{y}_1 - 2 \cdot A_2 \cdot \bar{y}_2}{A_1 - 2 \cdot A_2} = \frac{(21b^2)(1.5b) - 2(4b^2)(2b)}{21b^2 - 2(4b^2)}$$

$$\bar{Y} = \frac{31.5\, b^3 - 16 \cdot b^3}{13\, b^2} = \frac{15.5}{13} b = \boxed{1.19 \cdot b}$$

The anatomical center of gravity G of a person can be determined by using a scale and a rigid board having a uniform weight W_1 and length l. With the person's weight W known, the person lies down on the board and the scale reading P is recorded. From this, show how to calculate the location of the centroid $\bar{x}$. Discuss the best place l_1 for the smooth support at B in order to improve accuracy of this experiment.



$$(\Sigma M)_B = 0$$

$$\Rightarrow \; -P \cdot l_1 + W_{body} \cdot \bar{x} + W_{board} \cdot \Big[\underbrace{\tfrac{l}{2} - \left(l - \tfrac{l_1}{2}\right)}_{\left(\frac{l_1}{2} - \frac{l}{2}\right)} \Big] = 0$$

Solve for $\bar{x}$:

$$\bar{x} \cdot W_{body} = l_1 \cdot P + W_{board} \cdot \left(\frac{l}{2} - \frac{l_1}{2} \right)$$

$$\boxed{\bar{x} = l_1 \cdot \frac{P}{W_{body}} + \frac{1}{2} \cdot (l - l_1) \cdot \frac{W_{board}}{W_{body}}}$$

How to improve the accuracy of the experiment:

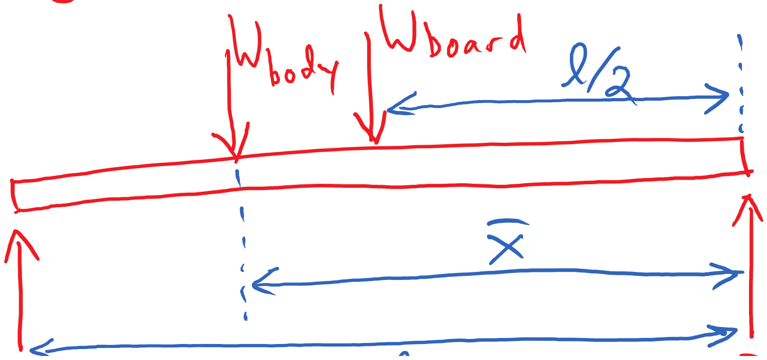
Goal: Find $\bar{x}$ based on the measurement of P.

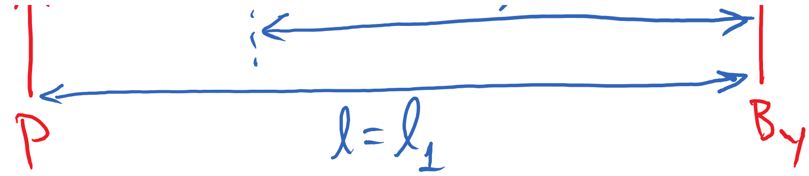
Sensitivity of $\bar{x}$ with respect to P is given by $\frac{\partial \bar{x}}{\partial P}$.

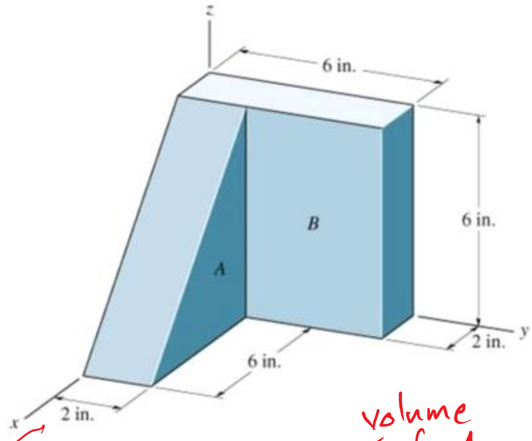
$$\Longrightarrow \quad \frac{\partial \bar{x}}{\partial P} = \frac{l_1}{W_{body}}$$

To increase the accuracy of $\bar{x}$, increase l_1. The greatest possible value of l_1 is l (the full length of the board.

$\rightarrow$




P
$l = l_1$
B_y



Two blocks of different materials are assembled as shown. The densities of the materials are:

$$\rho_A = 150 \text{ lb / ft}^3 \quad \text{and}$$
$$\rho_B = 400 \text{ lb / ft}^3.$$

Find: The center of gravity of this assembly.

M_i is mass of body i

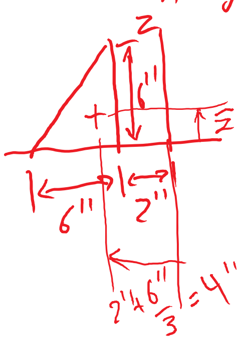
$$\boxed{\bar{x} \sum_i M_i = \sum \bar{x}_i M_i}$$

... likewise for $\bar{y}$ & $\bar{z}$

volume of A

$$M_A = \rho_A \cdot V_A = \left(150 \frac{lb}{ft^3}\right) \cdot \frac{1}{2}(6'')(6'')(2'')$$

thickness



$$\Rightarrow M_A = 3.125 \text{ lb}$$

$$\bar{x}_A = 2'' + \frac{6''}{3} = 4''$$

$$\bar{y}_A = 1''$$

$$\bar{z}_A = \frac{6''}{3} = 2''$$